微积分模拟卷答案.

1. $\frac{0}{0}$型 洛必达法则

考虑 $\frac{d}{dx}\int_0^x \frac{1}{t}(e^{xt}-1)dt = \frac{d}{dx}\int_0^x \frac{1}{xt}(e^{xt}-1)\,d(xt)$

$$\overset{令u=xt}{=} \frac{d}{dx}\int_0^{x^2} \frac{1}{u}(e^u-1)du$$

$$= 2x\cdot\frac{1}{x^2}(e^{x^2}-1) = \frac{2}{x}(e^{x^2}-1)$$

$\therefore \lim\limits_{x\to 0}\frac{\int_0^x \frac{1}{t}(e^{xt}-1)dt}{x^2} = \lim\limits_{x\to 0}\frac{\frac{2}{x}\cdot(e^{x^2}-1)}{2x} = 1$ #

2. 利用Taylor公式求极限

$n\to\infty$时 $f(a+\frac{1}{n}) = f(a) + f'(a)\cdot\frac{1}{n} + o(\frac{1}{n})$

$\therefore \lim\limits_{n\to\infty}\left[\frac{f(a+\frac{1}{n})}{f(a)}\right]^n = \lim\limits_{n\to\infty}\left[\frac{f(a)+f'(a)\cdot\frac{1}{n}+o(\frac{1}{n})}{f(a)}\right]^n$

$$= \lim_{n\to\infty}\left[1+\frac{f'(a)}{f(a)}\cdot\frac{1}{n}+o(\frac{1}{n})\right]^n$$

$$= e^{\frac{f'(a)}{f(a)}}$$ #

3. $\int_0^{\frac{\pi}{2}} f(\sin x)dx = \int_0^{\frac{\pi}{2}} f(\cos x)dx$ （令$t=\frac{\pi}{2}-x$ 即可）

$I = \int_0^{\frac{\pi}{2}}\frac{\cos^{2024}\alpha}{\sin^{2024}\alpha+\cos^{2024}\alpha}d\alpha = \int_0^{\frac{\pi}{2}}\frac{\sin^{2024}\alpha}{\sin^{2024}\alpha+\cos^{2024}\alpha}d\alpha$

$\therefore 2I = \int_0^{\frac{\pi}{2}} 1\,d\alpha = \frac{\pi}{2}$ $\quad\therefore I = \frac{\pi}{4}$ #

(1) 令 $a_n = x_n - \ln n = 1+\frac{1}{2}+\cdots+\frac{1}{n}-\ln n$

$\therefore a_{n+1}-a_n = \frac{1}{n+1}+\ln\frac{n}{n+1} = \frac{1}{n+1}+\ln(1-\frac{1}{n+1}) < \frac{1}{n+1}-\frac{1}{n+1} = 0$

$\therefore \{a_n\}$单调递减

因此$\{a_n\}$有上界 $a_1=1$，故要证 $\lim\limits_{n\to\infty}a_n$ 存在，只需证$\{a_n\}$有下界.

由 $\frac{1}{k} > \ln(1+\frac{1}{k}) = \ln\frac{k+1}{k}$

$\therefore a_n > \ln\frac{2}{1}+\ln\frac{3}{2}+\cdots+\ln\frac{n+1}{n}-\ln n = \ln\frac{n+1}{n} > 0$ 故0为下界

故 $\{a_n\}$极限存在 #

(2) 令 $\lim\limits_{n\to\infty} a_n = A$

∴ $\lim\limits_{n\to\infty} 1+\frac{1}{2}+\cdots+\frac{1}{n-1}-\ln(n-1) = A$ ①

$\lim\limits_{n\to\infty} 1+\frac{1}{2}+\cdots+\frac{1}{4n}-\ln 4n = A$ ②

②-①得 $\lim\limits_{n\to\infty}\left(\sum_{k=n+1}^{4n}\frac{1}{k}+\ln(n-1)-\ln 4n\right)=0$

∴ $\lim\limits_{n\to\infty}\sum_{k=n+1}^{4n}\frac{1}{k}=\lim\limits_{n\to\infty}\ln\frac{4n}{n-1}=\ln 4$ #

5. (1) $$\frac{dy}{dx}=\frac{\frac{dy}{dt}}{\frac{dx}{dt}}=\frac{a\cdot 3\sin^2 t\cdot\cos t}{-a\cdot 3\cos^2 t\cdot\sin t}=-\frac{\sin t}{\cos t}=-\tan t$$

(2)
$$V=\pi\int_0^{\pi} y^2(t)\cdot|d\,x(t)| = 3\pi a^3\int_0^{\pi}\sin^7 t\cos^2 t\,dt$$
$$=3\pi a^3\int_0^{\pi}\sin^7 t\,(1-\sin^2 t)\,dt$$
$$=6\pi a^3\int_0^{\frac{\pi}{2}}(\sin^7 t-\sin^9 t)\,dt=\frac{32}{105}\pi a^3$$

6. $f(x)=\arcsin x$

要求 $y^{(2024)}(0)$，只需令 $f(x)$ 在 $x=0$ 处 Taylor ~~展开~~即可

$$f'(x)=\frac{1}{\sqrt{1-x^2}}=\frac{1}{(1-x^2)^{\frac{1}{2}}}\triangleq h(x)$$

由 $h(x)=(1-x^2)^{-\frac{1}{2}}$

$h(x)$ 在 $x=0$ 处 Taylor 公式为：

$$h(x)=1+\frac{1}{2}x^2+\frac{1\cdot 3}{2\cdot 4}x^4+\cdots+\frac{(2n-1)!!}{(2n)!!}x^{2n}+o(x^{2n})$$

$y^{(2024)}(0)$ 即 $h^{(2023)}(0)=0$

7. $\int_1^{+\infty}\frac{\arctan x}{x^2}dx = \int_1^{+\infty} -\arctan x \, d(x^{-1})$

$= -\arctan x \cdot x^{-1}\Big|_1^{+\infty} - \int_1^{+\infty} x^{-1}\cdot(-\frac{1}{1+x^2})\,dx$

$= \frac{\pi}{4} + \int_1^{+\infty}(\frac{1}{x} - \frac{x}{x^2+1})\,dx$

$= \frac{\pi}{4} + \frac{1}{2}\ln 2$

8. $x\neq 0$ 时 $f'(x) = 2x\sin^2(\frac{1}{x}) + x^2\cdot 2\cos\frac{1}{x}\cdot(-\frac{1}{x^2})$

$= 2x\sin^2\frac{1}{x} - 2\cos\frac{1}{x}$

$x=0$ 时 $f'(0) = \lim\limits_{x\to 0}\frac{x^2\sin^2(\frac{1}{x}) - 0}{x} = \lim\limits_{x\to 0} x\sin^2\frac{1}{x} = 0$

$\therefore f'(x) = \begin{cases} 2x\sin^2\frac{1}{x} - 2\cos\frac{1}{x} & (x\neq 0) \\ 0 & (x=0) \end{cases}$

$x\neq 0$ 时 $f'(x)$ 连续

$x=0$ 处 由于 $\lim\limits_{x\to 0} f'(x) = \lim\limits_{x\to 0} 2x\sin^2\frac{1}{x} - 2\cos\frac{1}{x}$ 不存在

$\therefore$ $x=0$ 处 $f'(x)$ 不连续

9. $\int_0^1 x^2 f(x)\,dx = \int_0^1 f(x)\,d(\frac{1}{3}x^3) = \frac{1}{3}x^3 f(x)\Big|_0^1 - \int_0^1 \frac{1}{3}x^3 f'(x)\,dx$

$= \int_0^1 \frac{1}{3}x^3 e^{x^2}dx = -\frac{1}{3}\int_0^1 x^2 e^{x^2}\cdot\frac{1}{2}dx^2$

$= -\frac{1}{6}\int_0^1 t e^t\,dt = -\frac{1}{6}(t-1)e^t\Big|_0^1$

$= -\frac{1}{6}$

10、取 $x_0 = k_1 x_1 + k_2 x_2$ 将 $f(x_i)$ 在 $x = x_0$ 展开

$$f(x_1) = f(x_0) + f'(x_0)(x - x_0) + \frac{1}{2} f''(\xi)(x - x_0)^2$$

$$> f(x_0) + f'(x_0)(x_1 - x_0) \quad ①$$

$$f(x_2) > f(x_0) + f'(x_0)(x_2 - x_0) \quad ②$$

①$\cdot k_1$ + ②$\cdot k_2$ 得

$$k_1 f(x_1) + k_2 f(x_2) > f(x_0) + f'(x_0)(k_1 x_1 + k_2 x_2 - k_1 x_0 - k_2 x_0)$$

$$= f(x_0) + f'(x_0)(x_0 - x_0)$$

$$= f(x_0)$$

$\therefore k_1 f(x_1) + k_2 f(x_2) > f(x_0)$ #

11、要证 $\int_0^{\alpha} f(x)\,dx \geqslant \alpha \int_0^{\alpha} f(x)\,dx + \alpha \int_{\alpha}^{1} f(x)\,dx$

$$\Rightarrow \int_0^{\alpha} (1-\alpha) f(x)\,dx \geqslant \alpha \int_{\alpha}^{1} f(x)\,dx$$

即证 $\frac{1}{\alpha} \int_0^{\alpha} f(x)\,dx \geqslant \frac{1}{1-\alpha} \int_{\alpha}^{1} f(x)\,dx$

由于 $f(x)$ 单调不增

故 $\frac{1}{\alpha} \int_0^{\alpha} f(x)\,dx \geqslant f(\alpha) \geqslant \frac{1}{1-\alpha} \int_{\alpha}^{1} f(x)\,dx$

#

12. (1) $f(x) \le k + \int_a^x f(t)g(t)\,dt$

故 $$\frac{f(x)}{k+\int_a^x f(t)g(t)\,dt} \le 1$$

$$\frac{f(x)g(x)}{k+\int_a^x f(t)g(t)\,dt} \le g(x)$$

两边对x积分得

$$\int_a^s \frac{f(x)g(x)\,dx}{k+\int_a^x f(t)g(t)\,dt} \le \int_a^s g(x)\,dx$$

注意到: $f(x)g(x)\,dx = d\left(k+\int_a^x f(t)g(t)\,dt\right)$

故 左边 $= \ln\left[k+\int_a^x f(t)g(t)\,dt\right]\Big|_a^s$

所以 $\ln\left[k+\int_a^s f(t)g(t)\,dt\right] \le \ln k + \int_a^s g(x)\,dx$

又∵ $\ln f(s) \le$ 左侧

∴ $\ln f(s) \le \ln k + \int_a^s g(x)\,dx$

故 $f(s) \le k e^{\int_a^s g(x)dx}$ 证毕.

(2) 令 $k \to 0$

由于 $g(x)$ 在 $[a,b]$ 上有界

故 $k \cdot e^{\int_a^x g(t)dt} \to 0$

由于 $0 \le f(x) \le k \cdot e^{\int_a^x g(t)dt}$

∴ $k \to 0$ 时

$f(x)$ 只能 $= f(x) = 0$ (更严格地,本题可取极限为0的数列 $\{k_n\}$)